Reply to "Comment on 'Adaptive steady-state stabilization for nonlinear dynamical systems' "

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In a recent article [D. J. Braun, Phys. Rev. E **78**, 016213 (2008)], the author has proposed a controller to stabilize a steady-state of a nonlinear dynamical system without requiring explicit knowledge of the system dynamics. In the related Comment [W. Lin, Phys. Rev. E **81**, 038201 (2010)], Lin has criticized the analytical verification of the controller, and provided an alternative proof to verify its feasibility. While this proof is acknowledged here, we will show that the example used by Lin to demonstrate failure of the controller is a counterexample on the Comment itself.

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This article is a Reply to the Comment by Lin [1], where the analytical verification of the steady-state controller

$$\dot{x}_i = f_i(x) - k_i(x_i - y_i),$$

$$\dot{y}_i = \lambda_i(x_i - y_i),$$

$$\dot{k}_i = \alpha_i(x_i - y_i)^2, \qquad (1)$$

$i \in \{1,2,\ldots,n\}$, introduced in [2] is criticized. In the original article, it is claimed that the controller [Eq. (1)], initiated with $k_i(0)=0$, *can* stabilize the unstable equilibrium x^* of $\dot{x}_i = f_i(x)$, $f_i(x^*)=0$, without requiring any explicit knowledge of its position and even the original system dynamics $f_i(x)$.

In order to verify the controller, the author utilized the LaSalle invariance principle [3], application of which requires bounded response of the controlled system [Eq. (1)]. However, the fact that boundedness of the controlled response has not been assumed in [2] motivated Lin [1] to state: "However, there is a flaw in the analytical verification of this proposed controller. This flaw manifests that the LaSalle invariance principle [2] cannot be directly used in the proof, and that Braun's controller may be failed to stabilize some concrete systems."

Since verification of boundedness of the controlled response (in the context of the original article where the system dynamics is assumed unknown) is particularly hard, the consideration was restricted to systems with bounded uncontrolled response and stable steady-state estimator, $\lambda > 0$, under which conditions numerous systems have been numerically stabilized. Lin has commented that the mentioned two conditions *may* not ensure bounded controlled response and stabilization with Eq. (1). In addition, Lin showed that Eq. (1) can even stabilize systems with unbounded uncontrolled response if the control parameters are properly adjusted, which in some cases requires an unstable steady-state estimator, $\lambda < 0$, see Figs. 1 and 2. I acknowledge Lin's Comment, which points on some intrinsic features of the original idea and makes Eq. (1) more useful in chaos control.

However, in order to support the critique on the original article [2], Lin misinterpreted the results, and used numerical examples which are not related to the original idea. Below, it is specifically shown that neither the first example provided in the Comment to demonstrate failure of the controller, nor the second example which is presented to show successful stabilization (with particularly adjusted parameters) is correct by means of the original idea proposed by Braun [2]. Beyond this, other critical points in the comment are reviewed and shown to be irrelevant or incorrect. In this light, the critique by Lin does not provide an objective characterization of the controller proposed in [2] .

In the following we will *a priory* assume bounded response of Eq. (1), i.e., (x,y,k). While this assumption may

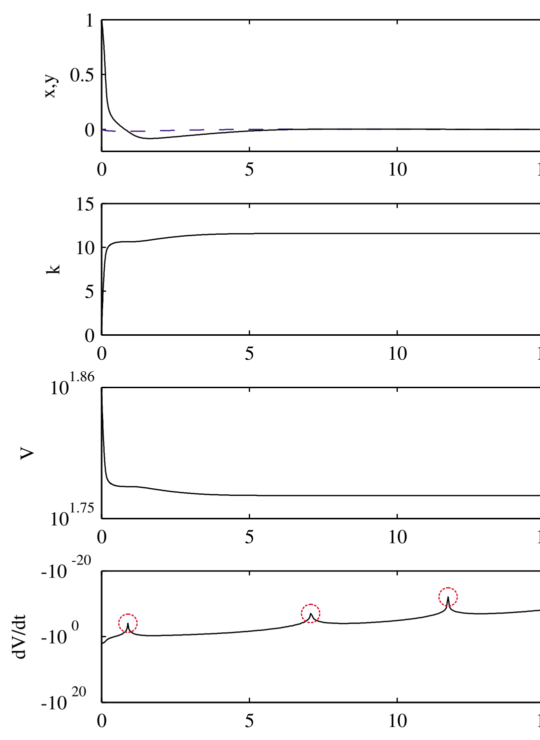

FIG. 1. (Color online) Stabilization of $\dot{x}=\sin(10x)$ (which has bounded uncontrolled response) to a locally unstable equilibrium $x^*=0$ using the steady-state controller [Eq. (1)]. The steady-state estimator y is depicted with dashed (blue) line. The control parameters are selected to be $\lambda=-0.1$ and $\alpha=100$, and the initial data for the controlled system is taken as $[x(0),y(0),k(0)]=[1,0,0]$. Functions, V and dV/dt are calculated with $L=120$. The numerical solver utilized in this example and along the paper is the *ode45* MATLAB solver with 10^{-6} relative and 10^{-10} absolute tolerance.

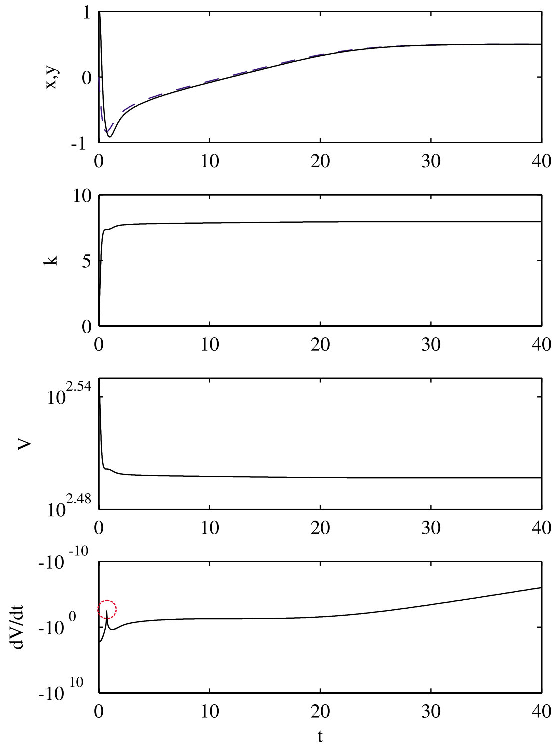

FIG. 2. (Color online) Stabilization of the system $\dot{x}=(x^3-1/8)$ (which has unbounded uncontrolled response) to its globally unstable equilibrium $x^*=1/2$, using the steady-state controller [Eq. (1)]. The steady-state estimator y is depicted with dashed (blue) line. The control parameters are selected to be $\lambda=-2$ and $\alpha=20$, and the initial data for the controlled system is taken as $[x(0),y(0),k(0)]=[1,0,0]$. Functions, V and dV/dt are calculated using $L=120$.

not hold for all systems, it was numerically verified for numerous dynamical systems (with bounded uncontrolled response), and as subsequently shown, it even holds for the example provided by Lin in his Comment which meant to demonstrate failure of the controller [Eq. (1)]. Before we further proceed and show that the mentioned example does not demonstrate failure, but on the contrary, it demonstrates successful stabilization, let us recall the Lyapunov candidate function (used in [2]),

$$V=\frac{1}{2}\sum_{i=1}^{n}(x_i-y_i)^2+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\alpha_i}(L-k_i)^2, \tag{2}$$

and its first derivative, calculated on Eq. (1),

$$\dot{V}=\sum_{i=1}^{n}(x_i-y_i)f_i(x)-(L+\lambda_i)(x_i-y_i)^2, \tag{3}$$

where L is a constant parameter. In order to ensure V to be a Lypunov function i.e., $\dot{V}\leq 0$, one should define a finite constant L as

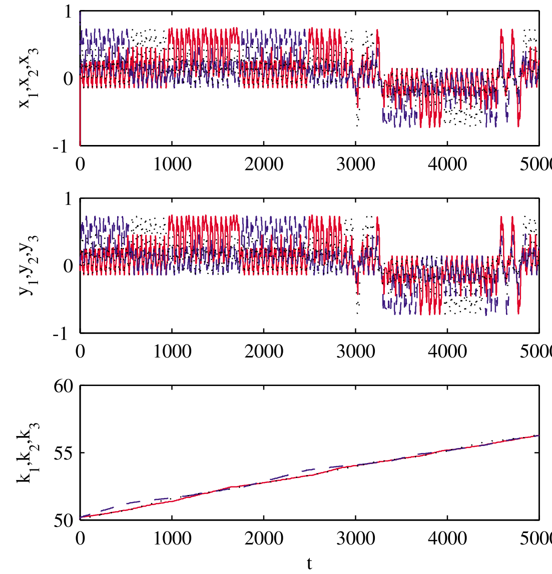

FIG. 3. (Color online) This figure depicts the response of Eq. (5) under the controller implemented in the preceding Comment [1] (where the corresponding result is reported in Fig. 2). The variables $(*)_{1,2,3}$ are depicted with full (red), dashed (blue), and dotted (black) lines. The control parameters are selected to be $\lambda_i=2$ and $\alpha_i=5$, $i\in\{1,2,3\}$ while the simulation is obtained with *large initial control gains* $k_i(0)=50$.

$$L>\sum_{i=1}^{n}(x_i-y_i)f_i(x)\Bigg/\sum_{i=1}^{n}(x_i-y_i)^2-\sum_{i=1}^{n}\lambda_i(x_i-y_i)^2\Bigg/\sum_{i=1}^{n}(x_i-y_i)^2. \tag{4}$$

Since x is (assumed) bounded, $f_i(x)<\infty$ holds for any continuous (or locally Lipschitz) vector field, and as such the right hand side of Eq. (4) is finite for any state of the system where $\Sigma_{i=1}^{n}(x_i-y_i)^2\neq 0$. For all these states, $L<\infty$ exist. If however $\Sigma_{i=1}^{n}(x_i-y_i)^2=0$, then $x_i=y_i$ for $\forall i$ and as such $\dot{V}=0$ regardless of the choice of L [in which case L does not need to satisfy Eq. (4)]. Note that $x_i=y_i\neq x_i^*$ for $\forall i$ is not an invariant state of the controlled system, and as such, although, $\dot{V}=0$, the system cannot be trapped in this non steady-state. A characteristic situation when $x_i=y_i\neq x_i^*$ for $\forall i$ can be identified with spikes in the dV/dt plots (denoted with circles), seen on Figs. 1 and 2, which scenario does not prevent stabilization.

While Lin has stated that $\dot{V}\leq 0$ is not a viable argument in the present context, it can be explicitly demonstrated that this argument holds through the stabilization process. To be specific, $\dot{V}\leq 0$, is verified on Figs. 1 and 2 and is also verified below on the example provided by Lin to demonstrate failure of Eq. (1).

Let us now recall the mentioned example,

$$\dot{x}_i=f_i(x)=-x_i+\sum_{j=1}^{3}w_{ij}\sin(10x_j) \tag{5}$$

where $i\in\{1,2,3\}$ and

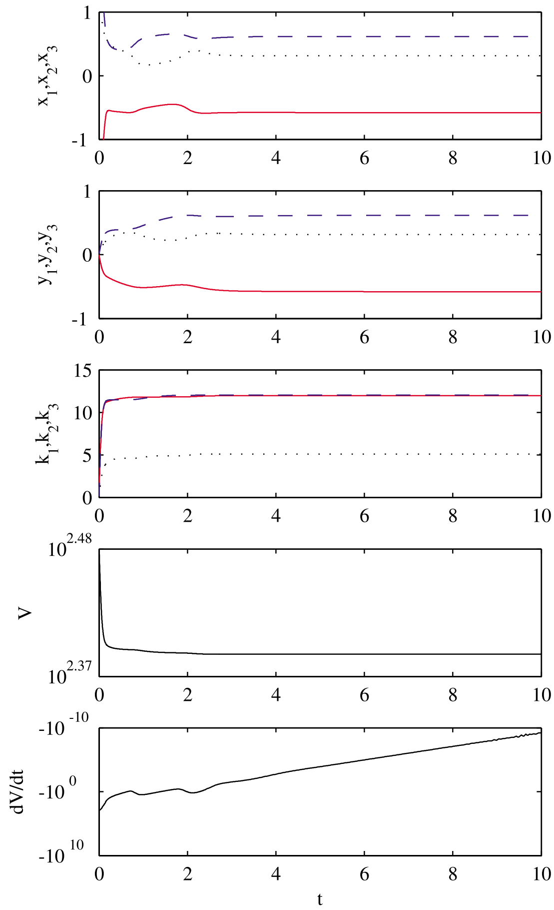

FIG. 4. (Color online) The dynamical behavior of the controlled system Eq. (1) shows a successful stabilization of Eq. (5). The variables $(*)_{1,2,3}$ are depicted with full (red), dashed (blue) and dotted (black) lines. The control parameters are selected to be $\lambda_i=2$ and $\alpha_i=50$, $i\in\{1,2,3\}$, and the initial data for the controlled system [Eq. (1)] is taken as $[x(0),y(0),k(0)]=[-2,2,1,0,0,0,0,0,0]$. Functions, V and dV/dt are calculated using $L=100$.

$$w_{ij}=\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}.$$

Based on the simulation result reported in the Comment on Fig. 2 (Fig. 3 here), Lin states: "Contrary to the analytical result given in [2], the stabilization is unsuccessful *with particularly selected parameters and initial values*."

In order to create such a result, Lin has utilized *large initial control gains* $k_i(0)=50$. Note however that as *developed* (and used) by Braun [2], the controller [Eq. (1)] needs to operate with *zero initial control gains*, indicating that application of Eq. (1) in the Comment does not respects the original idea. Accordingly, the numerical simulation in the Comment on Fig. 2 (Fig. 3 here) neither demonstrates failure, nor can a controller [Eq. (1)] initiated with nonzero gains be used to show successful stabilization as presented in the Comment on Fig. 3. *Both simulations presented by Lin are obtained by not respecting the original idea, and as such they have no relevance in the present context.*

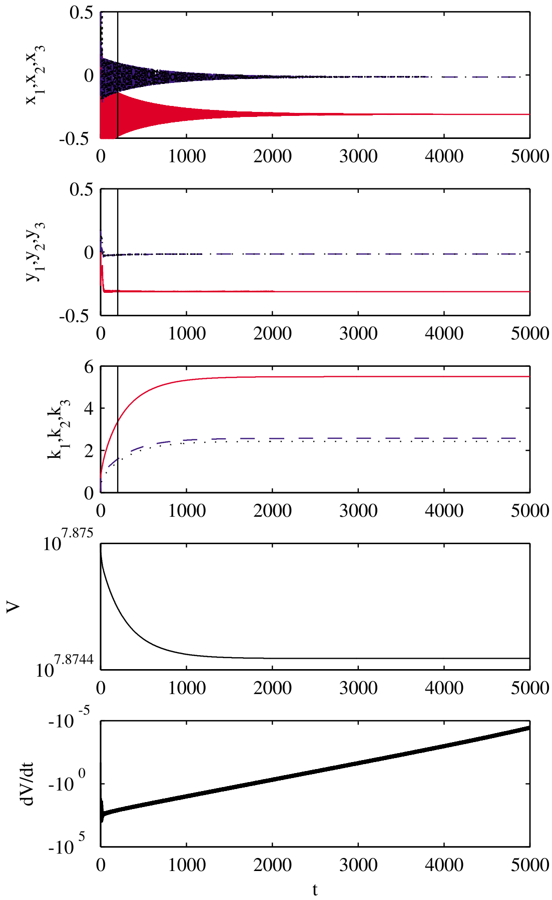

FIG. 5. (Color online) The dynamical behavior of the controlled system [Eq. (1)] shows a long transient in stabilization of Eq. (5). The variables $(*)_{1,2,3}$ are depicted with full (red), dashed (blue) and dotted (black) lines. The control parameters are selected to be $\lambda_i=0.2$ and $\alpha_i=0.5$, $i\in\{1,2,3\}$, and the initial data for controlled system [Eq. (1)] is taken as $[x(0),y(0),k(0)]=[-2,2,1,0,0,0,0,0,0]$. Functions, V and dV/dt are calculated using $L=5000$. The first 200 s of this simulation was presented by Lin, as a counterexample on [2], in the first accepted version of the Comment. While this example turned out not be a counterexample, it was excluded from the Comment, but is included here since it nicely demonstrates the effect of poorly selected control parameters.

Let us now show that the example [Eq. (5)] can be easily stabilized with the controller proposed by Braun, Fig. 4. For this purpose, the control parameters are adjusted here to obtain a short time transient response, $\lambda_i=2$, $\alpha_i=50$. This kind of parameter adjustment is a usual procedure for any controller rather than being a specific feature of the current one. Finding a *practical* parameter set (while not guaranteed to be always trivial), was neither involved for the examples reported in [2] nor was difficult for the current example introduced by Lin [1].

The numerical simulation depicted on Fig. 4 clarifies that an unstable equilibrium of Eq. (5) can be easily stabilized. Let us now further show that *this stabilization does not require specially selected control gains*. For this purpose, one can select $\lambda_i=0.2$ and $\alpha_i=0.5$ (instead of $\lambda_i=2$ and $\alpha_i=50$) and obtain stabilization after a long transient, Fig. 5. Note that poorly selected parameters (intentionally used here) generally result in a long time transient on any control system such that this feature is not some specific attribute of the proposed controller.

In addition to the numerical examples, Lin also introduced a theoretical condition on the feasibility of the proposed controller. This condition, denoted as ESL condition, requires that the equilibrium points of the uncontrolled system separately lie in $\mathbb{R}^n$. It is argued in the Comment that if the ESL condition is violated (in which case the uncontrolled system has continuous equilibria) the proposed controller *may* be infeasible. Now instead of demonstrating the practical implication of the ESL condition, Lin states: "To be candid, in practice, the convergence of $x(t)$ and $y(t)$ is always valid numerically, and then both $x(t)$ and $y(t)$ surely converge to the points embedded in the continuous equilibria."

In the Comment, Lin also argued that along the stabilization "each quantity $l_i = f_i(x)/(x_i - y_i)$ which is defined as the Lipschitz constant in [2], can be tremendously huge." This argument was then verified in simulation and used to support the claims on failure of the controller. There are two things which should be mentioned here. The first is that the original article nowhere defines l_i's as Lipschitz constants, while the second is that as long as x is bounded (which is verified through all stabilization examples) l_i can only tend to infinity (or be "tremendously huge" numerically) if $x_i = y_i$. This situation frequently happens during the stabilization process, *but it does not prevent stabilization* or leads to failure of the controller. For example, $x = y$ coincides with the dots on Figs. 1 and 2, which situation did not prevent stabilization.

In summary, while the alternative proof on the controller [Eq. (1)] presented in the Comment is acknowledged here, it is explicitly pointed out that the examples provided by Lin [1] do not demonstrate failure of Braun's controller [2]. Beyond this, other critical arguments stated in the Comment are reviewed and shown to be either irrelevant or incorrect.

[1] W. Lin, Phys. Rev. E **81**, 038201 (2010).

[2] D. J. Braun, Phys. Rev. E **78**, 016213 (2008).

[3] J. P. LaSalle, Proc. Natl. Acad. Sci. USA **46**, 363 (1960).